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A THEORY FOR TRANSVERSE VIBRATIONS OF THE TIMOSHENKO BEAM†

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A second-order variational formalism is presented for deriving the Timoshenko equation and boundary conditions consistent with it. Properties of the second high-frequency mode of vibrations predicted in the Timoshenko theory are investigated. It is shown that the frequencies of this mode depend non-analytically on a small parameter describing the influence of shear deformation on the transverse vibrations of the beam. When this parameter vanishes the frequencies of the second series do not return to the unperturbed values, but become infinite. Hence they cannot be predicted exactly, but the fact that they are being taken into account corrects and improves the values of the frequencies of the fundamental mode of vibrations. For these frequencies the Ostrogradskii energy of the Timoshenko beam turns out to be negative. The part played by the second mode of vibrations in the Timoshenko theory is discussed. A simple method for taking into account the effect of the deformation of the cross-section of the beam during the vibrations on its natural frequencies is suggested.

THE TIMOSHENKO equation describes transverse vibrations of an elastic beam (or rod) taking into account rotational inertia and transverse shear deformation [1]. For each spatial shape of the vibrations this equation gives two frequency values, i.e. it predicts two series of frequencies or two modes of vibration. Hence the Timoshenko model is treated as a two-mode approximation to the exact equations of the theory of elasticity [2]. Timoshenko himself only considered the low-frequency mode and did not discuss the second, high-frequency mode of vibration [1]. Despite the extensive literature on the Timoshenko theory [2] the role of the high-frequency mode remains obscure. This paper is an attempt to fill the gap.

1. SECOND-ORDER VARIATIONAL FORMALISM FOR THE TIMOSHENKO EQUATION

The Timoshenko equation [1] can be written in the form

$$y^{\cdot\cdot} + a_1 y'''' - a_2 y^{\cdot\cdot\prime\prime} + a_3 y^{\cdot\cdot\cdot\cdot} = 0 \tag{1.1}$$

$$a_1 = \frac{E_* I}{\rho F}, \quad a_2 = \frac{I}{F}\left(1 + \frac{E_*}{kG}\right), \quad a_3 = \frac{\rho I}{kFG}$$

Here $y(t, x)$ is the transverse displacement of the beam, the dot denotes differentiation with respect to time t, the prime denotes differentiation with respect to x, E_* is Young's modulus, G

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is the shear modulus, I is the moment of inertia of the cross-section of the beam with respect to the axis passing through the centre of gravity of this section and perpendicular to the plane of vibrations, F is the area of cross-section of the beam, ρ is the volume density of the beam material, and k is the shear coefficient (the ratio of the mean shear stress over the cross-section of the beam to the maximum shear stress).

Equation (1.1) can be obtained from the variational principle

$$\delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} L\,dt\,dx = 0 \tag{1.2}$$

if the Lagrange density has the form

$$L\,(\dot{y}, \ddot{y}, y', y'') = {}^1\!/_2\, [(\dot{y})^2 - a_1\,(y'')^2 - a_3(\ddot{y})^2 + a_2 \ddot{y} y''] \tag{1.3}$$

At times t_1 and t_2 the conditions

$$\delta y(t_1, x) = \delta y\,(t_2, x) = 0, \quad \delta \dot{y}\,(t_1, x) = \delta \dot{y}\,(t_2, x) = 0 \tag{1.4}$$

should be satisfied in (1.2).

One can verify that the Timoshenko equation (1.1) follows directly from (1.2)–(1.4). Moreover, from (1.2) we obtain the following conditions at the ends of the beam ($x = x_1,\ x_2$)

$$[a_1 y''' - {}^1\!/_2\, a_2 \ddot{y}'\,]\,\delta y = 0, \quad [a_1\, y'' - {}^1\!/_2\, a_2 \ddot{y}\,]\,\delta y' = 0 \tag{1.5}$$

At a hinged beam end we have $y = 0$ while y' can take any value. In this case, $\delta y = 0$ and $\delta y'$ is arbitrary. Using this and the second relation in (1.5) we find that at a hinged end

$$y = 0, \ y'' = 0 \tag{1.6}$$

If a beam end is rigidly embedded we have

$$y = 0, \ y' = 0 \tag{1.7}$$

At such an end δy and $\delta y'$ both vanish. Hence condition (1.5) is satisfied and there are no boundary conditions other than (1.7).

The Timoshenko equation has an integral of motion

$$E = \int_{x_1}^{x_2} (p_1 \dot{y} + p_2 \ddot{y} - L)\,dx \tag{1.8}$$

where p_1 and p_2 are the canonical momenta of the problem under consideration [3, 4]

$$p_1 = \partial L/\partial \dot{y} - \dot{p}_2, \quad p_2 = \partial L/\partial \ddot{y} \tag{1.9}$$

Using the explicit form of the Lagrange function (1.3) we obtain

$$p_1 = \dot{y} + a_3 \dddot{y} - {}^1\!/_2\, a_2 \dot{y}', \ p_2 = -a_3 \ddot{y} + {}^1\!/_2\, a_2 y'' \tag{1.10}$$

Formula (1.8) can now be rewritten as

$$E = {}^1\!/_2 \int_{x_1}^{x_2} \{(\dot{y})^2 + a_1 (y'')^2 - a_2 \dot{y}\, \dot{y}'' + a_3 [2 \dot{y}\, \dddot{y} - (\ddot{y})^2]\}\ dx \tag{1.11}$$

It is natural to call E the Ostrogradskii energy for the Timoshenko equation. First of all, E is a Noetherian conserved quantity corresponding to the invariance of the action

$$S = \int_{t_1}^{t_2} \int_{x_1}^{x_2} L \, dt \, dx \tag{1.12}$$

under time displacements $t \to t + \Delta t$, $\Delta t = \text{const}$ [5]. In theoretical physics it is conventional to call such a conserved quantity the energy [6]. In order for formula (1.11) to give a quantity with the dimensions of energy it is sufficient to multiply E by ρF. Secondly, E is the value of the Hamiltonian function at solutions of Eq. (1.11). In the problem under consideration the Hamiltonian should be constructed by the Ostrogradskii method, which generalizes the canonical formalism to dynamical systems with Lagrange functions depending on higher time derivatives of the coordinates [3].

The first two terms in (1.11) give the beam energy in the classical theory. The third term in (1.11), after integration by parts and using boundary conditions (1.6) or (1.7), can be replaced by the expression

$$\frac{a_2}{2} \int_{x_1}^{x_2} (\dot{y}')^2 \, dx$$

which is the rotational energy of the beam, because $\dot{y}'(t, x)$ is the rotational angular velocity of the transverse section of the rod in the Rayleigh approximation. As will be shown below (see Sec. 4) the quantity E is not the same as the mechanical energy of the Timoshenko beam and its precise physical meaning is still unclear.

2. FREQUENCY SPECTRUM OF THE TIMOSHENKO EQUATION

We shall construct a solution of Eq. (1.1) by the method of separation of variables

$$y(t, x) = e^{i\omega t} u(x) \tag{2.1}$$

Substituting (2.1) into (1.1) and into (1.6), (1.7) we obtain

$$a_1 u'''' + a_2 \omega^2 u'' + \omega^2 (a_3 \omega^2 - 1) u = 0 \tag{2.2}$$

and boundary conditions for the function $u(x)$. Below we will confine ourselves to considering a Timoshenko beam with hinged ends $x_2 = 0$, $x_2 = l$ (where l is the length of the beam)

$$u(0) = u''(0) = 0, \quad u(l) = u''(l) = 0 \tag{2.3}$$

The solution of Eq. (2.2) satisfying boundary conditions (2.3) has the form

$$u(x) = \sum_{n=1}^{\infty} C_n \sin \lambda_n x, \quad \lambda_n = \frac{n\pi}{l} \tag{2.4}$$

The frequency equation

$$a_3 \omega^4 - (1 + a_2 \lambda_n^2) \omega^2 + a_1 \lambda_n^4 = 0 \tag{2.5}$$

defines two series of real frequencies (two modes of vibration) [7]

$$(\omega_n^{\mp})^2 = [1 + a_2 \lambda_n^2 \mp \sqrt{(1 + a_2 \lambda_n^2)^2 - 4 a_1 a_3 \lambda_n^4}]/(2a_3), \quad n = 1, 2, \ldots \tag{2.6}$$

One can verify that all the frequencies are real. In reality, dissipative processes truncate the frequency spectrum for large n, but this question is beyond the scope of our considerations.

We shall now discuss the physical meaning of this frequency spectrum. The appearance of two frequency series in the Timoshenko theory is due to the last term in Eq. (1.1) containing the fourth time derivative of the unknown function and taking into account the influence of the transverse shear deformation on the beam vibrations. In the derivation of the Timoshenko equation this term, together with the Rayleigh term $-a_2 y^{\cdot\cdot\prime\prime}$, which takes into account the rotational inertia of an element of the beam during the vibrations, are considered to be small corrections to the classical Bernoulli–Euler theory, based on the equation

$$y^{\cdot\cdot} + a_1 y'''' = 0 \tag{2.7}$$

Consequently, in the Timoshenko theory it is natural to regard as physical only those frequencies which turn into frequencies of the Bernoulli–Euler equation (2.7) when the coefficients a_2 and a_3 vanish. Only the ω_n^- series satisfies this natural condition. The displacement frequencies ω_n^+ tend to infinity as $a_3 \to 0$.

In the Timoshenko equation (1.1) the coefficient a_3 is governed by the phenomenological parameter k. Hence the numerical value of a_3 can only be known with a certain degree of error. In a physically applicable phenomenological theory it is necessary for its predictions (in this case the frequencies) to depend smoothly on the phenomenological parameters. This condition is satisfied by the ω_n^- series

$$(\omega^-_{*n})^2 = 1 + \alpha_{3n} + O(\alpha_{3n}^2), \quad a_2 = 0 \tag{2.8}$$

$$\omega_{*n}^{\pm} = \omega_n^{\pm}/\Omega_n, \quad \Omega_n^2 = a_1 \lambda_n^4$$

$$\alpha_{3n} = a_3 \Omega_n^2 = a_1 a_3 \lambda_n^2 = \frac{E_*}{kG}\left(\frac{I}{F}\right)^2\left(\frac{n\pi}{l}\right)^4 \tag{2.9}$$

where Ω_n are the oscillation frequencies of the beam in the Bernoulli–Euler approximation.

In the asymptotic expansion (2.8) we have put $a_2 = 0$, which is quite acceptable in this case [8]. It is obvious that the dimensionless expansion parameter α_{3n} in (2.8) will only be small for the lower frequencies which, as has already been pointed out above, is the only region where the theory under consideration is applicable.

The second series ω_n^+ is singular as $a_3 \to 0$

$$(\omega^+_{*n})^2 \sim \alpha_{3n}^{-1}, \quad a_3 \to 0, \; a_2 = 0 \tag{2.10}$$

Hence small variations in the numerical values of a_3 lead to large errors in the predicted frequencies ω_n^+ as $a_3 \to 0$. One must also bear in mind that the frequencies ω_m^+ lie in the same domain as the higher frequencies of the fundamental series ω_n^-, $n > m$.

3. OSTROGRADSKII ENERGY OF THE TIMOSHENKO BEAM

The solutions of Eq. (1.1) corresponding to the first mode (y^-) and second mode (y^+) have the same spatial shape

$$y^{\mp} = C_n^{\mp} \sin(\omega_n^{\mp} t) \sin(n\pi x/l) \tag{3.1}$$

Substituting (3.1) into Eq. (1.11), using the frequency equation (2.5) and subsequently integrating with respect to dx we obtain

$$E_n^{\mp} = \tfrac{1}{2} M^2 (C_n^{\mp})^2 [\Omega_n^2 - a_3 (\omega_n^{\mp})^4], \quad n = 1, 2, \ldots; \; M = \rho F l \tag{3.2}$$

where Ω_n are the Bernoulli–Euler theory frequencies (2.9), M is the total mass of the beam, and $C_n^{\mp}$ are the vibration amplitudes. For formula (3.2) to have the correct dimensions we have introduced the factor ρF described in Sec. 1. In dimensionless variables (2.9), using the asymptotic expansions (2.8) and (2.10), we obtain

$$
\begin{aligned}
E_n^- &= \tfrac{1}{2} M^2 (C_n^-)^2 \Omega_n^2 (1 - \alpha_{3n} - 2\alpha_{3n}^2 - \dots) \\
E_n^+ &= \tfrac{1}{2} M^2 (C_n^+)^2 \Omega_n^2 (1 - \alpha_{3n}^{-2} - \dots), n = 1, 2, \dots, \quad \alpha_{3n} \to 0
\end{aligned} \tag{3.3}
$$

Thus, for sufficiently small values of the parameter α_{3n}, vibrations with frequencies ω_n^+ lead to negative values of the energy E_n^+. The energy has been defined so that the beam has zero energy at rest.

We will estimate the parameter α_{3n} in (2.9) numerically. For a beam of rectangular cross-section of height h and length l we have $I/F = h^2/12$, and hence

$$
\alpha_{3n} = \frac{E_*}{kG} \left(\frac{h}{l} \right)^4 \left(\frac{n\pi}{\sqrt{12}} \right)^4 \tag{3.4}
$$

Assuming $E_*/(kG) \sim 3$, $n = 1, 2, 3$ and considering sufficiently long beams $h/1 \sim 10^{-1}$, we obtain $\alpha_{3n} \sim 10^{-4}$. For such beams the energy E_n^+ is clearly negative and $E_n^- > 0$.

For short beams $h/1 \sim 1$ and large n the coefficient α_{3n} may turn out not to be small. But here one cannot use the asymptotic expansions (2.8), (2.10) and (3.3).

Difficulties with negative energies in theories involving higher derivatives are well known in elementary particle physics [9, 10]. But there is as yet no universal method for treating this problem in relativistically invariant field models [11, 12].

4. MECHANICAL ENERGY OF THE TIMOSHENKO BEAM

Unlike the field theories involving higher derivatives, that are considered in elementary particle physics, for the Timoshenko beam the well-defined concept of mechanical energy W exists. This is the sum of the kinetic energy T and the potential energy V of its separate elements [2]

$$
\begin{aligned}
& W = T + V \\
& T = \frac{\mu}{2} \int_0^l [(y^{\cdot})^2 + r^2 (\psi^{\cdot})^2] dx, \quad V = \frac{\mu c_0^2}{2} \int_0^l [r^2 \psi'^2 + \frac{kG}{E_*} (y' - \psi)^2] dx \\
& r^2 = I/F, \quad c_0 = \sqrt{E_*/\rho}
\end{aligned} \tag{4.1}
$$

Here as before $y(t, x)$ is the total transverse displacement of the beam, and $\psi(t, x)$ is the angle of inclination of the tangent to the bending curve, generated solely by the bending deformation of the beam, r is the radius of inertia of the transverse cross-section, and $\mu = \rho F$ is the density per unit length of the beam. There are no problems with any negativeness of this energy, because the functional W is positive definite. By varying the functional $T - V$ we obtain system of two coupled equations for $y(t, x)$ and $\psi(t, x)$

$$
\psi'' - \frac{1}{c_0^2} \psi^{\cdot\cdot} + \frac{kG}{E_* r^2} (y' - \psi) = 0, \quad y^{\cdot\cdot} - c_0^2 \frac{kG}{E_*} (y'' - \psi') = 0 \tag{4.2}
$$

In the case when a beam end is hinged, conditions (1.6) are supplemented with a boundary condition on the function $\psi(t, x)$

$$\psi' = 0 \tag{4.3}$$

Eliminating $\psi(t, x)$ from (4.2), we obtain the Timoshenko equation (1.1). One can similarly eliminate $y(t, x)$ from (4.2) and arrive at Eq. (1.1), only now for $\psi(t, x)$.†

According to (4.2), to the solution (3.1) for $y(t, x)$ there corresponds the following solution for $\psi(t, x)$

$$\psi^{\mp}(t, x) = B_n^{\mp} \sin(\omega_n^{\mp} t) \cos\left(n\pi \frac{x}{l}\right) \tag{4.4}$$

Substituting solution (4.4) into Eq. (4.2) we obtain the same frequency equation (2.5) and the ratio of the amplitudes $C_n^{\mp}$ and $B_n^{\mp}$

$$\frac{B_n^{\mp}}{C_n^{\mp}} = \frac{k_n^{\mp}}{l}, \quad k_n^{\mp} = n\pi\left[1 - \frac{E_*}{kG}\left(\frac{r}{l} n\pi\right)^2 (\omega_{*n}^{\mp})^2\right] \tag{4.5}$$

One can now find a relation between the energies of the second and first modes with the same number n. To fix our ideas we will assume that the amplitudes of these oscillations are equal: $C_n^- = C_n^+$. Substituting (3.1) and (4.4) into (4.1), we use the ratio (4.5) and obtain

$$\frac{W_n^+}{W_n^-} = \frac{1 + (k_n^+)^2 (r/l)^2}{1 + (k_n^-)^2 (r/l)^2} \left(\frac{\omega_n^+}{\omega_n^-}\right)^2 \tag{4.6}$$

Throughout the domain of practical applicability of the Timoshenko theory, for example, when $h/1 \leqslant 0.35$, the ratio (4.6) is clearly greater than 10^3. Thus the second high-frequency mode of vibration is practically unexcited in this case.

5. INFLUENCE OF VARIATIONS IN THE BEAM CROSS-SECTION DURING THE VIBRATION PROCESS ON ITS NATURAL FREQUENCIES

The Timoshenko theory is based on the explicit inclusion of macroscopic shear deformations experienced by each cross-section of the beam. One can go further down this road and attempt to include other forms of deformation. It is well known that under transverse beam vibrations the shape of its transverse section changes [13]. For simplicity we will consider a beam with a rectangular cross-section performing bending vibrations. Beams fibres lying below the neutral XZ plane (Fig. 1) undergo extension along the X axis which is accompanied by compression in the transverse direction. Fibres lying above the neutral plane experience longitudinal compression and therefore stretching in the transverse direction. As a result the beam cross-section acquires the shape shown in Fig. 2. The vertical sides of the rectangle are no longer parallel and become concentric beam segments.

This consideration is based on the exact solution of the three-dimensional equations of the theory of elasticity describing pure bending of a beam with a rectangular cross-section [14]. One can assume that the transverse fibres of the section of the beam shown in Fig. 2 acquire the shape of circular arcs of radius ρ_1. This radius is connected with the bending radius ρ of the beam by the well-known relation [13]

$$\rho_1 = \rho/\nu \tag{5.1}$$

†This feature was pointed out to the author by N. R. Shvets.

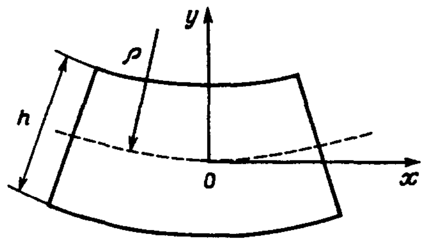

FIG. 1.

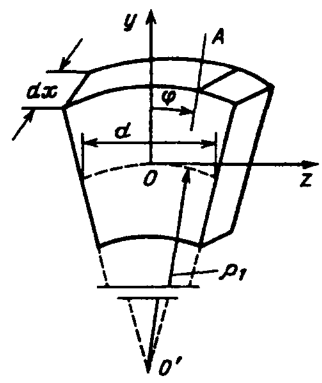

FIG. 2.

where ν is Poisson's ratio. As usual, we shall assume that the bending of the beam is given by a smooth curve so that one can set

$$1/\rho = \partial^2 y/\partial x^2 \tag{5.2}$$

where $y(t, x)$ is the transverse displacement of the beam in the XY plane.

The potential energy of the exact solution in the theory of elasticity describing pure bending of the beam is given by the same formula as in the Bernoulli–Euler theory [15]. It is therefore only necessary to take into account the contribution of this deformation to the kinetic energy of the vibrating beam.

We shall consider an element of the beam enclosed between two cross-sections at the points x and $x+dx$ (Fig. 2). The kinetic energy of this element is given by the formula

$$dT = \frac{1}{2}\,\rho dI \int_{-d/2}^{d/2} \dot{\varphi}^2 dz \tag{5.3}$$

where ρ is the material volume density, φ is the angle formed by a radial section (the line $O'A$ in Fig. 2) with the Y axis and dI is the moment of inertia of this element of the section with the XY plane

$$dI = h^3 dx/12 \tag{5.4}$$

From Fig. 2 we have

$$\varphi = z/\rho_1 = \nu z/\rho = \nu z y'' \tag{5.5}$$

Using (5.4) and (5.5), formula (5.3) can be reduced to the following form

$$dT = {}^1\!/_2\, \nu^2 \rho I_z r_y^2 (\dot{y}'')^2 dx, \quad I_z = h^3 d/12, \; r_y = I_y/F = d^2/12 \tag{5.6}$$

where I_z is the moment of inertia of the beam cross-section about the Z axis, d is the width of the beam, and r_y is the radius of inertia of the beam cross-section about the Y axis.

For the frequencies we will find the correction due to this deformation. For simplicity we will first use the original Bernoulli–Euler–Rayleigh theory. Taking into account correction (5.6) we can write the Lagrange function L in the form

$$L = T - V = {}^1\!/_2\, \rho F \int_0^l (y^{\cdot})^2\, dx + {}^1\!/_2\, \rho I_z \int_0^l (y^{\cdot\,\prime})^2 dx +$$

$$+ \frac{\nu^2}{2}\, \rho I_z r_y^2 \int_0^l (y^{\cdot\,\prime\prime})^2 dx - {}^1\!/_2\, E_* I_z \int_0^l (y^{\prime\prime})^2 dx \tag{5.7}$$

Here l is, as before, the length of the beam. The beam vibration equation

$$y^{\cdot\cdot} + c_0^2\, r_z^2\, y^{\prime\prime\prime\prime} - r_z^2\, y^{\cdot\cdot\prime\prime} + \nu^2\, r_z^2\, r_y^2\, y^{\cdot\cdot\prime\prime\prime\prime} = 0, \quad r_z = I_z/F \tag{5.8}$$

follows from (5.7).

For simplicity we shall confine ourselves to the case of a hinge-supported beam for which boundary conditions (1.6) hold. Separating the variables in (5.8)

$$y(t, x) = A \sin(\omega_n t + \epsilon) \sin(n\pi x/l) \tag{5.9}$$

we obtain the following frequency equation

$$\omega_{*n}^2 = \delta_{zn}^2\, [1 - \omega_{*n}^2\, (1 + \nu^2 \delta_{yn}^2)], \quad \omega_{*n} = \omega_n \frac{l}{c_0 n\pi}, \tag{5.10}$$

$$\delta_{yn} = n\pi \frac{r_y}{l} = \frac{n\pi}{\sqrt{12}} \frac{d}{l}, \quad \delta_{zn} = n\pi \frac{r_z}{l} = \frac{n\pi}{\sqrt{12}} \frac{h}{l} \tag{5.11}$$

where ω_{*n} are dimensionless frequencies and δ_{yn}, δ_{zn} are dimensionless characteristics of the beam cross-section.

Comparing (5.10) with the Bernoulli–Euler–Rayleigh theory, we note the appearance of the new factor in parentheses, which can only amplify the Rayleigh correction. As might have been expected, this is only noticeable for high frequencies (large n) and wide beams. It should be noted that within this elementary theory formulae (5.10) already show the dependence of natural frequencies of the beam on Poisson's ratio ν.

Similar derivations are easily performed within the Timoshenko theory. If correction (5.6) is taken into account when calculating the kinetic energy (formula (4.1)) only the first equation in (4.2) is changed. It acquires the form

$$\psi'' - \frac{1}{c_0^2}\, \psi^{\cdot\cdot} + \frac{\xi}{r_z^2}\, (y' - \psi) + \frac{\nu^2}{c_0^2}\, \frac{d^2}{12}\, \psi^{\cdot\cdot\prime\prime} = 0, \quad \xi = kG/E_*$$

Consider once again a hinge-supported beam. For $y(t, x)$ we have solution (5.9), and $\psi(t, x)$ is given by the similar formula

$$\psi(t, x) = B \sin(\omega_n t + \epsilon) \cos(n\pi x/l)$$

The frequency equation has the form

$$\omega_{*n}^4\, (1 + \nu^2 \delta_{yn}^2) - \omega_{*n}^2\, [1 + \xi + \xi\, (\delta_{zn}^{-2} + \nu^2 \delta_{yn}^2)] + \xi = 0$$

Here ω_{*n} are, as before, the dimensionless frequencies. The terms proportional to ν^2 are new compared to (2.5). As in the previous case, one can verify that these corrections are only significant for high frequencies and wide beams.

Remarks. 1. With the aid of the frequency equation (2.5) one can show that the ratio (4.5) never becomes infinite. Consequently, in this case there are no solutions for which the total beam deflection $y(t, x)$ is zero but the shear displacement $\psi(t, x)$ is non-zero (notwithstanding the assertion in [8]).

2. In [16] formula (61.84) contains an error k_{ns} should be replaced by k_{ns}^2.

3. There is no sense in talking about the possibility of resonant observation of the second series of frequencies in the Timoshenko theory, although this is sometimes done (see, e.g., [17]).

4. Explaining the physical content of the concept of the Ostrogradskii energy of the oscillating Timoshenko beam is of undoubted interest. Like any integral of the dynamical equations, it should play an important role in the general analysis of the solutions.

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